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DOI 10.1515/tmj-2017-0051

Conformable fractional Hermite-Hadamard inequalities via preinvex functions

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Abstract

The aim of this paper is to obtain some new refinements of Hermite-Hadamard type inequalities via conformable fractional integrals. The class of functions used for deriving the inequalities have the preinvexity property. We also discuss some special cases.

2010 Mathematics Subject Classification. **26D15.** 26A51, 26A33
Keywords. Convex, preinvex, fractional, conformable, Hermite-Hadamard.

1 Introduction and preliminaries

During the last century theory of convexity had been developed rapidly. Several new classes of classical convexity have been proposed in the literature, see [2]. Hanson [5] has introduced the class of differentiable invex functions. Mititelu [8] introduced the concept of invex set, as follows.
Let K_η be a nonempty set in $\mathbb{R}$. Let $f : K_\eta \to \mathbb{R}$ be a continuous function and let $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function.

Definition 1.1 ([8])**.** A set $K_\eta \in \mathbb{R}$ is said to be invex with respect to η, if

$$x + t\eta(y, x) \in K_\eta, \quad \forall x, y \in K_\eta, t \in [0, 1]. \tag{1.1}$$

The concept of invex set K_η is also referred as η-connected set.

Remark 1.2. If $\eta(y, x) = y - x$, then invexity of set K_η reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(y, x) = y - x$, but the converse is not necessarily true.

Recently Noor et al. [13] introduced the notion of h-preinvex functions as follows:

Definition 1.3 ([13])**.** Let $h : J \to \mathbb{R}$ where $(0, 1) \subseteq J$ be an interval in $\mathbb{R}$, and let K_η be an invex set with respect to $\eta(., .)$. A function $f : K_\eta \to \mathbb{R}$ is called h-preinvex with respect to $\eta(., .)$, if

$$f(x + t\eta(y, x)) \leq h(1 - t)f(x) + h(t)f(y), \quad x, y \in K_\eta, t \in (0, 1).$$

If above inequality is reversed, then f is said to be h-preincave with respect to $\eta(., .)$.

Tbilisi Mathematical Journal 10(4) (2017), pp. 129–141.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 10 April 2016.
Accepted for publication: 06 September 2017.

Remark 1.4. Noor et al. [13] have observed that for suitable choice of function $h(.)$ the class of h-preinvex functions not only includes classical preinvex functions introduced in [20], but also other new classes of preinvex functions such as: Breckner type of s-preinvex functions, Godunova-Levi preinvex functions and P-preinvex functions respectively. Noor et al. [12] introduced another class of preinvex functions which is called as Godunova-Levin type of s-preinvex functions and noticed that this class also included in the class of h-preinvex functions. For $\eta(y,x) = y - x$ the class of h-preinvex functions reduces to the class of h-convex functions introduced and studied by Varosanec [19]. Thus it is worth to mention here that the class of h-preinvex is quite unifying one.

Remark 1.5. In this paper function $\eta(.,.) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is supposed to have the following property:

$$\eta(y + t_1\eta(x,y), y + t_2\eta(x,y)) = (t_1 - t_2)\eta(x,y), \quad \forall t_1, t_2 \in [0,1], t_1 \leq t_2. \tag{1.2}$$

In this case the following consequences hold:

1. If $t_1 = t_2 = 0$ then (1.2) implies that $\eta(y,y) = 0$ for all $y \in \mathbb{R}$.
2. If $t_1 = 0$ and $t_2 = t > 0$ then $\eta(y, y + t\eta(x,y)) = -t\eta(x,y)$ for all $x, y \in \mathbb{R}$. This is the first requirement of Condition C introduced in [9].
3. If $\eta(x,y) > 0$ for some $(x,y) \in \mathbb{R}$ then $\eta(y, y + t\eta(x,y)) \leq 0$ for all $t \in [0,1]$. It means that property (1.2) implies that function η has not constant sign on $\mathbb{R}\times\mathbb{R}$.

The relationship between theory of convexity and theory of inequalities has inspired many researchers and consequently many new inequalities have been obtained via convexity property of the functions. One of the most famous inequality in this regard is Hermite-Hadamard's inequality, which is equivalent property of the convexity. This inequality reads as follows:

Theorem 1.6. Let $f : I = [a,b] \to \mathbb{R}$ be a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

Hermite-Hadamard like inequality via h-preinvex functions reads as follows:

Theorem 1.7. Let $f : I_\eta = [a, a+\eta(b,a)] \to (0,\infty)$ be an h-preinvex function with $\eta(b,a) > 0$, $h(\frac{1}{2}) \neq 0$. If $\eta(.,.)$ satisfies (1.2), then we have

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \leq [f(a)+f(b)]\int_0^1 h(t)\mathrm{d}t.$$

Sarikaya et al. [15] extended Hermite-Hadamard's inequality via fractional integrals. Recently Set et al. [17] have given a new generalization of Hermite-Hadamard's inequality via conformable fractional integrals. For some more on inequalities via conformable fractional integrals, see [17, 18]. We now give some preliminary concepts and results which will be helpful in obtaining our main results.

Definition 1.8 ([7])**.** Let $f \in L_1[a,b]$. Then Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} x^{\alpha-1} dx,$$

is the Gamma function.

The well-known Euler Beta function is defined as:

$$B(a,b) = \int\limits_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t, \quad a,b > 0.$$

The relationship between Gamma and Euler Beta functions is given as:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

The incomplete Beta function is defined as:

$$B_x(a,b) = \int\limits_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t, \quad x \in [0,1].$$

Note that for $x = 1$ the incomplete Beta functions reduce to classical Beta function.
Abdeljawad [1] have given the definition of left and right conformable fractional integrals of any order $\alpha > 0$ as:

Definition 1.9 ([1])**.** Let $\alpha \in (n, n+1]$ and $\beta = \alpha - n$. Then the left and right conformable fractional integrals starting at a of order α as defined by

$$J_{\alpha}^{a} f(t) = \frac{1}{n!} \int\limits_a^t (t-x)^n (x-a)^{\beta-1} f(x)\mathrm{d}x,$$

and

$$J_{\alpha}^{b} f(t) = \frac{1}{n!} \int\limits_t^b (x-t)^n (b-x)^{\beta-1} f(x)\mathrm{d}x.$$

Note that if $\alpha = n+1$ then $\beta = 1$ where $n = 0, 1, 2, \dots$.
It is worth to mention here that

$$B(a,b) = B_t(a,b) + B_{1-t}(a,b).$$

$$\begin{aligned} B_x(a+1,b) &= \frac{aB_x(a,b) - (\frac{1}{2})^{a+b}}{a+b}; \\ B_x(a,b+1) &= \frac{bB_x(a,b) - (\frac{1}{2})^{a+b}}{a+b}. \end{aligned}$$

Also using the properties of incomplete Beta functions, we have

$$\begin{aligned} &B_{1-t}(n+1, \alpha-n) - B_t(n+1, \alpha-n) \\ &= \int_0^{1-t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x - \int_0^t x^n(1-x)^{\alpha-n-1}\mathrm{d}x \\ &= \int_t^{1-t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x, \quad t \in \left[0, \frac{1}{2}\right]. \end{aligned} \tag{1.3}$$

Also

$$\begin{aligned} &B_t(n+1, \alpha-n) - B_{1-t}(n+1, \alpha-n) \\ &= \int_0^{t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x - \int_0^{1-t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x \\ &= \int_{1-t}^{t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x, \quad t \in \left[\frac{1}{2}, 1\right]. \end{aligned} \tag{1.4}$$

2 Main results

In this section, we derive our main results.

Theorem 2.1. Let $f : I_\eta \to \mathbb{R}$ be a h-preinvex function such that $f \in L_1[a, a+\eta(b,a)]$, $\eta(b,a) > 0$ where $\eta(.,.)$ satisfies (1.2) and $h(\frac{1}{2}) \neq 0$, then

$$\begin{aligned} &\frac{\Gamma(\alpha-n)}{h(\frac{1}{2})\Gamma(\alpha+1)} f\left(\frac{2a+\eta(b,a)}{2}\right) \\ &\leq \frac{1}{\eta^\alpha(b,a)}\left\{J_\alpha^{a+\eta(b,a)}f(a) + J_\alpha^a f(a+\eta(b,a))\right\} \\ &\leq \frac{1}{n!}\{f(a)+f(b)\}\int_0^1 t^n(1-t)^{\alpha-n-1}[h(1-t)+h(t)]\mathrm{d}t. \end{aligned}$$

Proof. Let $u, v \in [a, a+\eta(b,a)]$ and f be preinvex on $[a,b]$ such that $\eta(.,.)$ satisfies (1.2), then

$$f\left(\frac{u+v}{2}\right) \le \frac{f(u)+f(v)}{2}.$$

Let $u = a + t\eta(b,a)$ and $v = a + (1-t)\eta(b,a)$, then

$$\frac{1}{h(\frac{1}{2})} f\left(\frac{2a+\eta(b,a)}{2}\right) \le f(a+t\eta(b,a)) + f(a+(1-t)\eta(b,a)).$$

Multiplying both sides of above inequality with $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$ and then integrating it with respect to t on $[0,1]$, we have

$$\frac{1}{n!h(\frac{1}{2})} f\left(\frac{2a+\eta(b,a)}{2}\right) \int_0^1 t^n(1-t)^{\alpha-n-1}\mathrm{d}t$$
$$\le \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1} f(a+t\eta(b,a))\mathrm{d}t + \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1} f(a+(1-t)\eta(b,a))\mathrm{d}t.$$

This implies

$$\frac{1}{n!h(\frac{1}{2})} f\left(\frac{2a+\eta(b,a)}{2}\right) \frac{\Gamma(n+1)\Gamma(\alpha-n)}{\Gamma(\alpha+1)}$$
$$\le \frac{1}{n!}\int_a^{a+\eta(b,a)} \left(\frac{x-a}{\eta(b,a)}\right)^n \left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\alpha-n-1} f(x)\mathrm{d}x$$
$$+ \frac{1}{n!}\int_a^{a+\eta(b,a)} \left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^n \left(\frac{x-a}{\eta(b,a)}\right)^{\alpha-n-1} f(x)\mathrm{d}x.$$

This implies

$$\frac{\Gamma(\alpha-n)}{h(\frac{1}{2})\Gamma(\alpha+1)} f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta^\alpha(b,a)}\left\{J_\alpha^{a+\eta(b,a)} f(a) + J_\alpha^a f(a+\eta(b,a))\right\}, \tag{2.1}$$

which is the left part of the inequality. Now we prove the second part of the inequality. Since it is given that f is preinvex function, then

$$f(a+t\eta(b,a)) + f(a+(1-t)\eta(b,a)) \le [f(a)+f(b)][h(1-t)+h(t)].$$

Multiplying both sides of above inequality with $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$ and then integrating it with respect to t on $[0,1]$, we have

$$\frac{1}{\eta^\alpha(b,a)}\left\{J_\alpha^{a+\eta(b,a)} f(a) + J_\alpha^a f(a+\eta(b,a))\right\}$$
$$\le \frac{1}{n!}\{f(a)+f(b)\}\int_0^1 t^n(1-t)^{\alpha-n-1}[h(1-t)+h(t)]\mathrm{d}t. \tag{2.2}$$

On summation of inequalities (2.1) and (2.2) the proof is complete. Q.E.D.

We now discuss some special cases of Theorem 2.1.

I. If $h(t) = t$ in Theorem 2.1, then, we have following new result for classical preinvex functions.

Corollary 2.2. Let $f : I_\eta \to \mathbb{R}$ be a preinvex function such that $f \in L_1[a, a+\eta(b,a)]$ and $\eta(b,a) > 0$ where $\eta(.,.)$ satisfies (1.2), then

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \\ \leq \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha-n)\eta^\alpha(b,a)}\left\{J_\alpha^{a+\eta(b,a)} f(a) + J_\alpha^a f(a+\eta(b,a))\right\} \leq \frac{f(a)+f(b)}{2}.$$

II. If $h(t) = t^s$ in Theorem 2.1, then, we have following new result for Breckner type of s-preinvex functions.

Corollary 2.3. Let $f : I_\eta \to \mathbb{R}$ be Breckner type of s-preinvex function where $s \in (0,1]$ such that $f \in L_1[a, a+\eta(b,a)]$ and $\eta(b,a) > 0$ where $\eta(.,.)$ satisfies (1.2), then

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)} f\left(\frac{2a+\eta(b,a)}{2}\right) \\ \leq \frac{1}{2^s\eta^\alpha(b,a)}\left\{J_\alpha^{a+\eta(b,a)} f(a) + J_\alpha^a f(a+\eta(b,a))\right\} \\ \leq \left\{\frac{B(n+s+1,\alpha-n) + B(n+1,\alpha-n+s)}{n!}\right\}\frac{f(a)+f(b)}{2^s}.$$

III. If $h(t) = t^{-s}$ in Theorem 2.1, then, we have following new result for Godunova-Levin type of s-preinvex functions.

Corollary 2.4. Let $f : I_\eta \to \mathbb{R}$ be Godunova-Levin type of s-preinvex function where $s \in [0,1]$ such that $f \in L_1[a, a+\eta(b,a)]$ and $\eta(b,a) > 0$ where $\eta(.,.)$ satisfies (1.2), then

$$\frac{\Gamma(\alpha-n)}{2^s\Gamma(\alpha+1)} f\left(\frac{2a+\eta(b,a)}{2}\right) \\ \leq \frac{1}{\eta^\alpha(b,a)}\left\{J_\alpha^{a+\eta(b,a)} f(a) + J_\alpha^a f(a+\eta(b,a))\right\} \\ \leq \left\{\frac{B(n-s+1,\alpha-n) + B(n+1,\alpha-n-s)}{n!}\right\}[f(a)+f(b)].$$

IV. If $h(t) = 1$ in Theorem 2.1, then, we have following new result for P-preinvex functions.

Corollary 2.5. Let $f : I_\eta \to \mathbb{R}$ be P-preinvex function such that $f \in L_1[a, a+\eta(b,a)]$ and $\eta(b,a) > 0$ where $\eta(.,.)$ satisfies (1.2), then

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \\ \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)\eta^\alpha(b,a)}\left\{J_\alpha^{a+\eta(b,a)} f(a) + J_\alpha^a f(a+\eta(b,a))\right\} \leq 2[f(a)+f(b)].$$

The following result is an auxiliary result and it plays a key role in obtaining next results.

Lemma 2.6. Let $f : I_\eta \to \mathbb{R}$ be a differentiable function on I_η°. If $f' \in L[a, a+\eta(b,a)]$, then, the following equality holds:

$$T_f(a,b;n;\eta;\alpha;B;J) = \frac{\eta(b,a)}{2}\left[\int_0^1 \left[B_{1-t}(n+1,\alpha-n) - B_t(n+1,\alpha-n)\right] f'(a+(1-t)\eta(b,a))\mathrm{d}t\right],$$

where

$$T_f(a,b;n;\eta;\alpha;B;J) = B(n+1,\alpha-n)\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right) - \frac{n!}{2\eta^\alpha(b,a)}\left[J_\alpha^{[a+\eta(b,a)]} f(a) + J_\alpha^a f(a+\eta(b,a))\right]$$

Proof. Let

$$I = \int_0^1 \left[B_{1-t}(n+1,\alpha-n) - B_t(n+1,\alpha-n)\right] f'(a+(1-t)\eta(b,a))\mathrm{d}t. \tag{2.3}$$

Now, suppose

$$\begin{aligned}
I_1 &= \int_0^1 B_{1-t}(n+1,\alpha-n) f'(a+(1-t)\eta(b,a))\mathrm{d}t \\
&= \int_0^1 \left(\int_0^{1-t} x^n (1-x)^{\alpha-n-1}\mathrm{d}x\right) f'(a+(1-t)\eta(b,a))\mathrm{d}t \\
&= \left(\int_0^1 x^n(1-x)^{\alpha-n-1}\mathrm{d}x\right)\frac{f(a+\eta(b,a))}{\eta(b,a)} \\
&\qquad - \frac{1}{\eta(b,a)}\int_0^1 (1-t)^n t^{\alpha-n-1} f(a+(1-t)\eta(b,a))\mathrm{d}t \\
&= \left(\int_0^1 x^n(1-x)^{\alpha-n-1}\mathrm{d}x\right)\frac{f(a+\eta(b,a))}{\eta(b,a)} \\
&\qquad - \frac{1}{\eta^2(b,a)}\int_a^{a+\eta(b,a)} \left(\frac{x-a}{\eta(b,a)}\right)^n \left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\alpha-n-1} f(x)\mathrm{d}x \\
&= B(n+1,\alpha-n)\frac{f(a+\eta(b,a))}{\eta(b,a)} - \frac{n!}{\eta^{\alpha+1}(b,a)} J_\alpha^{[a+\eta(b,a)]} f(a).
\end{aligned} \tag{2.4}$$

Also

$$I_2 = \int_0^1 B_t(n+1, \alpha - n) f'(a + (1-t)\eta(b,a))\mathrm{d}t$$

$$= -B(n+1, \alpha - n)\frac{f(a)}{\eta(b,a)} + \frac{n!}{\eta^{\alpha+1}(b,a)} J_\alpha^a f(a + \eta(b,a)). \tag{2.5}$$

On summation of (2.3), (2.4) and (2.5) and then multiplying by $\frac{\eta(b,a)}{2}$ completes the proof. Q.E.D.

Theorem 2.7. Let $f : I_\eta \to \mathbb{R}$ be a differentiable function on I_η° such that $f' \in L[a, a+\eta(b,a)]$ and $\eta(b,a) > 0$. If $|f'|$ is h-preinvex function, then the following inequality holds:

$$|T_f(a,b;n;\eta;\alpha;B;J)| \leq \frac{\eta(b,a)}{2}\left[(A+C)|f'(a)| + (B+D)|f'(b)|\right],$$

where

$$A := \int_0^{\frac{1}{2}} \left\{B_{1-t}(n+1,\alpha-n) - B_t(n+1,\alpha-n)\right\} h(t)\mathrm{d}t; \tag{2.6}$$

$$B := \int_0^{\frac{1}{2}} \left\{B_{1-t}(n+1,\alpha-n) - B_t(n+1,\alpha-n)\right\} h(1-t)\mathrm{d}t; \tag{2.7}$$

$$C := \int_{\frac{1}{2}}^1 \left\{B_t(n+1,\alpha-n) - B_{1-t}(n+1,\alpha-n)\right\} h(t)\mathrm{d}t; \tag{2.8}$$

and

$$D := \int_{\frac{1}{2}}^1 \left\{B_t(n+1,\alpha-n) - B_{1-t}(n+1,\alpha-n)\right\} h(1-t)\mathrm{d}t, \tag{2.9}$$

respectively, where $B(a,b)$, $B_t(a,b)$ are Euler Beta and incomplete Beta functions respectively and $\alpha \in (n, n+1]$, where $n \in 0, 1, 2 \ldots$.

Proof. Using Lemma 2.6, property of modulus and the fact that $|f'|$ is h-preinvex function, we have

$$\begin{aligned}
&|T_f(a,b;n;\eta;\alpha;B;J)| \\
&= \left|\frac{\eta(b,a)}{2}\left[\int_0^1 \left[B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right] f'(a+(1-t)\eta(b,a))\mathrm{d}t\right]\right| \\
&\leq \frac{\eta(b,a)}{2}\left[\int_0^1 |B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,|f'(a+(1-t)\eta(b,a))|\,\mathrm{d}t\right] \\
&= \frac{\eta(b,a)}{2}\left[\int_0^{\frac{1}{2}} \{B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\}\,|f'(a+(1-t)\eta(b,a))|\,\mathrm{d}t\right] \\
&\quad + \frac{\eta(b,a)}{2}\left[\int_{\frac{1}{2}}^1 \{B_t(n+1,\alpha-n)-B_{1-t}(n+1,\alpha-n)\}\,|f'(a+(1-t)\eta(b,a))|\,\mathrm{d}t\right] \\
&\leq \frac{\eta(b,a)}{2}\left[\int_0^{\frac{1}{2}} \{B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\}\,[h(t)|f'(a)|+h(1-t)|f'(b)|]\mathrm{d}t\right] \\
&\quad + \frac{\eta(b,a)}{2}\left[\int_{\frac{1}{2}}^1 \{B_t(n+1,\alpha-n)-B_{1-t}(n+1,\alpha-n)\}\,[h(t)|f'(a)|+h(1-t)|f'(b)|]\mathrm{d}t\right] \\
&= \frac{\eta(b,a)}{2}\left[(A+C)|f'(a)|+(B+D)|f'(b)|\right].
\end{aligned}$$

This completes the proof. Q.E.D.

We now discuss some special cases of Theorem 2.7.

I. If $h(t)=t$ in Theorem 2.7, then, we have following new result for classical preinvex functions.

Corollary 2.8. Let $f: I_\eta \to \mathbb{R}$ be a differentiable function on I_η° such that $f' \in L[a,a+\eta(b,a)]$ and $\eta(b,a)>0$. If $|f'|$ is preinvex function, then the following inequality holds:

$$\begin{aligned}
&|T_f(a,b;n;\eta;\alpha;B;J)| \\
&\leq \frac{\eta(b,a)}{2}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]\Big\{B_{\frac{1}{2}}(\alpha-n+2,n+1)-B_{\frac{1}{2}}(n+1,\alpha-n+2) \\
&\quad +B_{\frac{1}{2}}(n+3,\alpha-n)-B_{\frac{1}{2}}(\alpha-n,n+3)+B(n+1,\alpha-n)\Big\}.
\end{aligned}$$

II. If $h(t)=t^s$ in Theorem 2.7, then, we have following new result for Breckner type of s-preinvex functions.

Corollary 2.9. Let $f: I_\eta \to \mathbb{R}$ be a differentiable function on I_η° such that $f' \in L[a,a+\eta(b,a)]$ and $\eta(b,a)>0$. If $|f'|$ is Breckner type of s-preinvex function, where $s\in(0,1]$, then the following inequality holds:

$$\begin{aligned}&|T_f(a,b;n;\eta;\alpha;B;J)|\\&\leq \frac{\eta(b,a)}{2}\left[\frac{|f'(a)|+|f'(b)|}{s+1}\right]\Big\{B_{\frac{1}{2}}(\alpha-n+s+1,n+1)-B_{\frac{1}{2}}(n+1,\alpha-n+s+1)\\&\quad+B_{\frac{1}{2}}(n+s+2,\alpha-n)-B_{\frac{1}{2}}(\alpha-n,n+s+2)+B(n+1,\alpha-n)\Big\}.\end{aligned}$$

III. If $h(t)=t^{-s}$ in Theorem 2.7, then, we have following new result for Godunova-Levin type of s-preinvex functions.

Corollary 2.10. Let $f:I_\eta\to\mathbb{R}$ be a differentiable function on I_η° such that $f'\in L[a,a+\eta(b,a)]$ and $\eta(b,a)>0$. If $|f'|$ is Godunova-Levin type of s-preinvex function, where $s\in[0,1]$, then the following inequality holds:

$$\begin{aligned}&|T_f(a,b;n;\eta;\alpha;B;J)|\\&\leq \frac{\eta(b,a)}{2}\left[\frac{|f'(a)|+|f'(b)|}{1-s}\right]\Big\{B_{\frac{1}{2}}(\alpha-n-s+1,n+1)-B_{\frac{1}{2}}(n+1,\alpha-n-s+1)\\&\quad+B_{\frac{1}{2}}(n-s+2,\alpha-n)-B_{\frac{1}{2}}(\alpha-n,n-s+2)+B(n+1,\alpha-n)\Big\}.\end{aligned}$$

IV. If $h(t)=1$ in Theorem 2.7, then, we have following new result for P-preinvex functions.

Corollary 2.11. Let $f:I_\eta\to\mathbb{R}$ be a differentiable function on I_η° such that $f'\in L[a,a+\eta(b,a)]$ and $\eta(b,a)>0$. If $|f'|$ is P-preinvex function, then the following inequality holds:

$$\begin{aligned}&|T_f(a,b;n;\eta;\alpha;B;J)|\\&\leq \frac{\eta(b,a)}{2}\left[|f'(a)|+|f'(b)|\right]\Big\{B_{\frac{1}{2}}(\alpha-n+1,n+1)-B_{\frac{1}{2}}(n+1,\alpha-n+1)\\&\quad+B_{\frac{1}{2}}(n+2,\alpha-n)-B_{\frac{1}{2}}(\alpha-n,n+2)+B(n+1,\alpha-n)\Big\}.\end{aligned}$$

Theorem 2.12. Let $f:I_\eta\to\mathbb{R}$ be a differentiable function on I_η° such that $f'\in L[a,a+\eta(b,a)]$ and $\eta(b,a)>0$. If $|f'|^q$ is h-preinvex function, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$\begin{aligned}&|T_f(a,b;n;\eta;\alpha;B;J)|\\&\leq \frac{\eta(b,a)}{2}\left[2\int_0^{\frac{1}{2}}\left(\int_t^{1-t}x^n(1-x)^{\alpha-n-1}\mathrm{d}x\right)^p\right]^{\frac{1}{p}}\left[(|f'(a)|^q+|f'(b)|^q)\int_0^1 h(t)\mathrm{d}t\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. Using Lemma 2.6, Holder's inequality and the fact that $|f'|^q$ is h-preinvex function, we have

$$\begin{aligned}
&|T_f(a,b;n;\eta;\alpha;B;J)| \\
&= \left|\frac{\eta(b,a)}{2}\left[\int_0^1 \left[B_{1-t}(n+1,\alpha-n) - B_t(n+1,\alpha-n)\right] f'(a+(1-t)\eta(b,a))\mathrm{d}t\right]\right| \\
&\le \frac{\eta(b,a)}{2}\left[\int_0^1 |B_{1-t}(n+1,\alpha-n) - B_t(n+1,\alpha-n)|\,|f'(a+(1-t)\eta(b,a))|\,\mathrm{d}t\right] \\
&\le \frac{\eta(b,a)}{2}\left[\int_0^1 |B_{1-t}(n+1,\alpha-n) - B_t(n+1,\alpha-n)|^p\,\mathrm{d}t\right]^{\frac{1}{p}} \\
&\qquad\qquad \times \left[\int_0^1 |f'(a+(1-t)\eta(b,a))|^q\,\mathrm{d}t\right]^{\frac{1}{q}} \\
&\le \frac{\eta(b,a)}{2}\left[2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x\right)^p\right]^{\frac{1}{p}}\left[(|f'(a)|^q+|f'(b)|^q)\int_0^1 h(t)\mathrm{d}t\right]^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. Q.E.D.

We now discuss some special cases of Theorem 2.12.

I. If $h(t) = t$ in Theorem 2.12, then, we have following new result for classical preinvex functions.

Corollary 2.13. Let $f : I_\eta \to \mathbb{R}$ be a differentiable function on I_η° such that $f' \in L[a, a+\eta(b,a)]$ and $\eta(b,a) > 0$. If $|f'|^q$ is preinvex function, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$\begin{aligned}
&|T_f(a,b;n;\eta;\alpha;B;J)| \\
&\le \frac{\eta(b,a)}{2}\left[2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x\right)^p\right]^{\frac{1}{p}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

II. If $h(t) = t^s$ in Theorem 2.12, then, we have following new result for Breckner type of s-preinvex functions.

Corollary 2.14. Let $f : I_\eta \to \mathbb{R}$ be a differentiable function on I_η° such that $f' \in L[a, a+\eta(b,a)]$ and $\eta(b,a) > 0$. If $|f'|^q$ is Breckner type of s-preinvex function, where $s \in (0,1], p > 1, \frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$|T_f(a,b;n;\eta;\alpha;B;J)| \le \frac{\eta(b,a)}{2}\left[2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x\right)^p\right]^{\frac{1}{p}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right]^{\frac{1}{q}}.$$

III. If $h(t)=t^{-s}$ in Theorem 2.12, then, we have following new result for Godunova-Levin type of s-preinvex functions.

Corollary 2.15. Let $f: I_\eta \to \mathbb{R}$ be a differentiable function on I_η° such that $f' \in L[a, a+\eta(b,a)]$ and $\eta(b,a)>0$. If $|f'|^q$ is Godunova-Levin type of s-preinvex function, where $s\in[0,1], p>1, \frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$|T_f(a,b;n;\eta;\alpha;B;J)| \le \frac{\eta(b,a)}{2}\left[2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x\right)^p\right]^{\frac{1}{p}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{1-s}\right]^{\frac{1}{q}}.$$

IV. If $h(t)=1$ in Theorem 2.12, then, we have following new result for P-preinvex functions.

Corollary 2.16. Let $f: I_\eta \to \mathbb{R}$ be a differentiable function on I_η° such that $f' \in L[a, a+\eta(b,a)]$ and $\eta(b,a)>0$. If $|f'|^q$ is P-preinvex function, where $p>1, \frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$|T_f(a,b;n;\eta;\alpha;B;J)| \le \frac{\eta(b,a)}{2}\left[2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x\right)^p\right]^{\frac{1}{p}}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}.$$

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